Foundations of Physics (2022) 52:61
https://doi.org/10.1007/s10701-022-00578-6

Derivation from Bloch Equation to von Neumann Equation to Schrödinger–Pauli Equation

Lihong V. Wang[1]

Received: 9 July 2021 / Accepted: 8 May 2022

Abstract

The transition from classical physics to quantum mechanics has been mysterious. Here, we mathematically derive the space-independent von Neumann equation for electron spin from the classical Bloch equation. Subsequently, the space-independent Schrödinger–Pauli equation is derived in both the quantum mechanical and recently developed co-quantum dynamic frameworks.

Keywords Bloch equation · von Neumann equation · Liouville–von Neumann equation · Schrödinger–Pauli equation · Schrödinger equation · Landau–Lifshitz–Gilbert equation · Electron spin

1 Introduction

The Schrödinger equation, as a postulate, is a corner stone in quantum mechanics. The transition from classical physics to quantum mechanics, however, remains a mystery. Various approaches to obtaining the time-dependent Schrödinger equation have been investigated [1–6]. Recently, Schleich et al. generalized the Hamilton–Jacobi equation to reach the Schrödinger equation [7]. Most notably, Feynman used the path integral to attain the same equation [8].

We investigate the transition from classical physics to the space-independent Schrödinger–Pauli equation for electron spin. In classical electrodynamics, the motion of the magnetic dipole moment of an electron is governed by the Bloch equation. Majorana stated that both the classical and the quantum–mechanical treatments on spin flip of atoms moving in a magnetic quadrupole field require integration of the same differential equations [9, 10]. It is known that the space-independent Schrödinger–Pauli equation or von Neumann equation (also known as the

✉ Lihong V. Wang
LVW@caltech.edu

1 Department of Electrical Engineering, Andrew and Peggy Cherng Department of Medical Engineering, California Institute of Technology, 1200 E. California Blvd., MC 138-78, Pasadena, CA 91125, USA

Liouville–von Neumann equation) for a unitary two-level system can be converted to the Bloch equation or its analog [11–13]. However, the inverse conversion that would complete the two-way transitions has not been found in the literature.

Here, the classical Bloch equation for electron spin is mathematically converted to the space-independent von Neumann equation for a pure state of a two-level spin system. Subsequently, the space-independent Schrödinger–Pauli equation is derived in both frameworks of quantum mechanics and recently developed co-quantum dynamics (CQD, see Appendix). Therefore, the inverse conversion is shown, and the two-way transitions for a pure state of electron spin between the classical Bloch equation and the space-independent Schrödinger–Pauli equation are established.

2 Derivation from Bloch Equation to Space-Independent von Neumann Equation

We start with the classical Bloch equation for an electron in a magnetic field,

$$\frac{1}{\gamma}\frac{d}{dt}\vec{\mu} = \vec{\mu} \times \vec{B}, \tag{1}$$

where $\vec{\mu}$ denotes the magnetic dipole moment of the electron,γ the gyromagnetic ratio, t time, and $\vec{B}$ the magnetic flux density. Substitution of $\vec{\mu} = \frac{\hbar}{2}\gamma\hat{\mu}$ yields

$$\frac{\hbar}{2}\frac{d}{dt}\hat{\mu} = \frac{\hbar}{2}\gamma\hat{\mu} \times \vec{B}. \tag{2}$$

Here, $\hbar$ denotes the reduced Planck constant, $\frac{\hbar}{2}$ the spin angular momentum of the electron, and caret a unit vector. The unit vector is expressed as

$$\hat{\mu} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \tag{3}$$

where θ and ϕ denote the polar and azimuthal angles.

We now resort to the Pauli vector,

$$\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}, \tag{4}$$

as a mathematical tool, which transforms real-space (x, y, z) vectors and operations using complex numbers. The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

Note that the Pauli matrices, related to quaternions, are applied beyond quantum mechanics.

Multiplying both sides of the Bloch equation (Eq. 2) by $\vec{\sigma}$ from the right yields

$$\frac{\hbar}{2}\left(\frac{d}{dt}\hat{\mu}\right)\cdot\vec{\sigma} = \frac{\hbar}{2}\gamma\left(\hat{\mu}\times\vec{B}\right)\cdot\vec{\sigma}. \tag{6}$$

Merging the factors on the left side and splitting the right side produces

$$\frac{\hbar}{2}\frac{d}{dt}(\hat{\mu}\cdot\vec{\sigma}) = \frac{\hbar}{2}\gamma\left[\frac{1}{2}\left(\hat{\mu}\times\vec{B}\right)\cdot\vec{\sigma} - \frac{1}{2}\left(\vec{B}\times\hat{\mu}\right)\cdot\vec{\sigma}\right]. \tag{7}$$

Applying the following mathematical identity—known as the Pauli vector identity—for any two vectors $\vec{a}$ and $\vec{b}$,

$$(\vec{a}\cdot\vec{\sigma})\left(\vec{b}\cdot\vec{\sigma}\right) = \left(\vec{a}\cdot\vec{b}\right)\sigma_0 + i\left(\vec{a}\times\vec{b}\right)\cdot\vec{\sigma}, \tag{8}$$

we obtain

$$(\hat{\mu}\cdot\vec{\sigma})\left(\vec{B}\cdot\vec{\sigma}\right) = \left(\hat{\mu}\cdot\vec{B}\right)\sigma_0 + i\left(\hat{\mu}\times\vec{B}\right)\cdot\vec{\sigma} \tag{9}$$

and

$$\left(\vec{B}\cdot\vec{\sigma}\right)(\hat{\mu}\cdot\vec{\sigma}) = \left(\vec{B}\cdot\hat{\mu}\right)\sigma_0 + i\left(\vec{B}\times\hat{\mu}\right)\cdot\vec{\sigma}. \tag{10}$$

Subtraction of the above two equations produces

$$(\hat{\mu}\cdot\vec{\sigma})\left(\vec{B}\cdot\vec{\sigma}\right) - \left(\vec{B}\cdot\vec{\sigma}\right)(\hat{\mu}\cdot\vec{\sigma}) = i\left[\left(\hat{\mu}\times\vec{B}\right)\cdot\vec{\sigma} - \left(\vec{B}\times\hat{\mu}\right)\cdot\vec{\sigma}\right]. \tag{11}$$

Substituting into Eq. 7 yields

$$i\hbar\frac{d}{dt}\left(\frac{1}{2}\hat{\mu}\cdot\vec{\sigma}\right) = \frac{1}{2}\hbar\gamma\left[\left(\frac{1}{2}\hat{\mu}\cdot\vec{\sigma}\right)\left(\vec{B}\cdot\vec{\sigma}\right) - \left(\vec{B}\cdot\vec{\sigma}\right)\left(\frac{1}{2}\hat{\mu}\cdot\vec{\sigma}\right)\right]. \tag{12}$$

We define

$$\rho = \frac{1}{2}\left(\hat{\mu}\cdot\vec{\sigma} + \sigma_0\right). \tag{13}$$

Substituting Eqs. 3–5 into Eq. 13 produces

$$\rho = \begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}\exp(-i\phi) \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}\exp(i\phi) & \sin^2\frac{\theta}{2} \end{pmatrix}, \tag{14}$$

which reproduces the familiar quantum mechanical density matrix for a pure state of electron spin. One may express the density matrix using the state vector in outer-product form as follows:

$$\rho = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}\exp(-i\phi) \end{pmatrix}. \tag{15}$$

We next define

$$H = -\frac{1}{2}\hbar\gamma\vec{B}\cdot\vec{\sigma}, \tag{16}$$

which reproduces the familiar quantum mechanical Hamiltonian for electron spin. Note that $\frac{1}{2}\hbar\gamma$ is the magnitude of the magnetic dipole moment.

Substituting Eqs. 13 and 16 into Eq. 12 yields

$$i\hbar\frac{d}{dt}\Big(\rho - \frac{1}{2}\sigma_0\Big) = H\Big(\rho - \frac{1}{2}\sigma_0\Big) - \Big(\rho - \frac{1}{2}\sigma_0\Big)H. \tag{17}$$

Eliminating the identity matrix σ_0 (Eq. 5) produces the von Neumann equation,

$$i\hbar\frac{d}{dt}\rho = H\rho - \rho H = [H, \rho]. \tag{18}$$

Therefore, the classical Bloch equation is mathematically converted to the space-independent von Neumann equation for a pure state of electron spin.

3 Quantum Mechanical Derivation from Von Neumann Equation to Space-Independent Schrödinger–Pauli Equation

While the Schrödinger equation naturally evolves to the von Neumann equation [14], the inverse process holds for a pure state [15]. The quantum mechanical density matrix for a pure state is given by

$$\rho = |\hat{\mu}\rangle\langle\hat{\mu}|. \tag{19}$$

The *ket* and *bra* vectors [14] are given by

$$|\hat{\mu}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix} \tag{20}$$

and

$$\langle\hat{\mu}| = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}\exp(-i\phi) \end{pmatrix}. \tag{21}$$

Substituting Eq. 19 into 18 yields

$$i\hbar\frac{d}{dt}\big(|\hat{\mu}\rangle\langle\hat{\mu}|\big) = \big[H, |\hat{\mu}\rangle\langle\hat{\mu}|\big]. \tag{22}$$

Expansion produces

$$\left(i\hbar\frac{d}{dt}|\hat{\mu}\rangle\right)\langle\hat{\mu}| + |\hat{\mu}\rangle\left(i\hbar\frac{d}{dt}\langle\hat{\mu}|\right) = \left(H|\hat{\mu}\rangle\right)\langle\hat{\mu}| - |\hat{\mu}\rangle\left(\langle\hat{\mu}|H\right). \tag{23}$$

Combining terms results in

$$\left(i\hbar\frac{d}{dt}|\hat{\mu}\rangle - H|\hat{\mu}\rangle\right)\langle\hat{\mu}| = -|\hat{\mu}\rangle\left(i\hbar\frac{d}{dt}\langle\hat{\mu}| + \langle\hat{\mu}|H\right). \tag{24}$$

Multiplying both sides by $|\hat{\mu}\rangle$ from the right and using $\langle\hat{\mu}|\hat{\mu}\rangle = 1$ yields

$$\left(i\hbar\frac{d}{dt}|\hat{\mu}\rangle - H|\hat{\mu}\rangle\right) = -|\hat{\mu}\rangle\left(i\hbar\frac{d}{dt}\langle\hat{\mu}| + \langle\hat{\mu}|H\right)|\hat{\mu}\rangle. \tag{25}$$

Because the *bra* and *ket* vectors [14] originate from independent realizations, presumably implied in the omitted derivations by Wieser [15], both sides must vanish for the equation to hold. Therefore, we reach

$$i\hbar\frac{d}{dt}|\hat{\mu}\rangle = H|\hat{\mu}\rangle, \tag{26}$$

which is the space-independent Schrödinger–Pauli equation.

Conversely, we now assume that the *bra* and *ket* vectors originate from identical realizations and examine the consequence. Differentiating $\langle\hat{\mu}|\hat{\mu}\rangle = 1$ yields

$$\left(\frac{d}{dt}\langle\hat{\mu}|\right)|\hat{\mu}\rangle + \langle\hat{\mu}|\left(\frac{d}{dt}|\hat{\mu}\rangle\right) = 0. \tag{27}$$

Substituting into Eq. 25 and using Eq. 19 produces

$$(1-\rho)\left(i\hbar\frac{d}{dt}|\hat{\mu}\rangle - H|\hat{\mu}\rangle\right) = 0. \tag{28}$$

If the matrix, $1-\rho$, is full rank, multiplying both sides from the left by the inverse matrix uniquely yields the space-independent Schrödinger–Pauli equation. However, from Eq. 14, we have the determinant, $|1-\rho| = 0$; thus, the matrix is singular. Consequently, the space-independent Schrödinger–Pauli equation can be reached as only a sufficient but not necessary condition. To reach the Schrödinger–Pauli equation as a sufficient and necessary condition, we assume that the *bra* and *ket* vectors [14] originate from independent realizations.

4 CQD Derivation from Von Neumann Equation to Space-Independent Schrödinger–Pauli Equation

In Sect. 3, the independent realizations are implicit in the *bra* and *ket* vectors (Eqs. 20 and 21). On the basis of explicit independent realizations in CQD (see Appendix), we repeat the derivation. Henceforth, subscripted e and n denote the electron and nucleus, respectively, in the same atom.

The CQD pre-collapse state function is denoted by $|\widehat{\mu}_e©\widehat{\mu}_n\rangle$, where the co-quantum, $\widehat{\mu}_n$, is prefixed with © for clarity. $|\widehat{\mu}_e©\widehat{\mu}_n\rangle$ represents the principal quantum, $\widehat{\mu}_e$, accompanied with $\widehat{\mu}_n$. For a given $\widehat{\mu}_e$, the CQD prediction expressions for two independent realizations are written in dual spaces as follows:

$$|\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle = C_{1+}\left(\widehat{\mu}_e, \widehat{\mu}_{n1}\right)|+z\rangle + C_{1-}\left(\widehat{\mu}_e, \widehat{\mu}_{n1}\right)\exp(+i\phi_e)|-z\rangle \tag{29}$$

and

$$\langle\widehat{\mu}_e©\widehat{\mu}_{n2}| = C_{2+}\left(\widehat{\mu}_e, \widehat{\mu}_{n2}\right)\langle+z| + C_{2-}\left(\widehat{\mu}_e, \widehat{\mu}_{n2}\right)\exp\left(-i\phi_e\right)\langle-z|. \tag{30}$$

Numbers in subscripts denote independent realizations. Each binary coefficient represents either one or zero according to the CQD branching condition.

As shown in Sect. 2, the von Neumann equation was derived without ensemble averaging. Thus, we start with the following pre-averaging density operator:

$$\rho_0 = |\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\langle\widehat{\mu}_e©\widehat{\mu}_{n2}|. \tag{31}$$

To illustrate the parallelism with Sect. 3, we keep the original wording as much as possible below.

The von Neumann equation becomes

$$i\hbar\frac{d}{dt}\left(|\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\langle\widehat{\mu}_e©\widehat{\mu}_{n2}|\right) = \left[H, |\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\langle\widehat{\mu}_e©\widehat{\mu}_{n2}|\right], \tag{32}$$

where H is assumed to be shared by the two realizations. Expansion yields

$$\begin{aligned} &i\hbar\left(\frac{d}{dt}|\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\right)\langle\widehat{\mu}_e©\widehat{\mu}_{n2}| + |\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle i\hbar\frac{d}{dt}\langle\widehat{\mu}_e©\widehat{\mu}_{n2}| \\ &= \left(H|\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\right)\langle\widehat{\mu}_e©\widehat{\mu}_{n2}| - |\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\left(\langle\widehat{\mu}_e©\widehat{\mu}_{n2}|H\right). \end{aligned} \tag{33}$$

Rearranging terms produces

$$\left[\left(i\hbar\frac{d}{dt} - H\right)|\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\right]\langle\widehat{\mu}_e©\widehat{\mu}_{n2}| = |\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\left(-i\hbar\frac{d}{dt}\langle\widehat{\mu}_e©\widehat{\mu}_{n2}| - \langle\widehat{\mu}_e©\widehat{\mu}_{n2}|H\right). \tag{34}$$

Multiplying both sides by $|\widehat{\mu}_e©\widehat{\mu}_{n2}\rangle$ from the right gives

$$\begin{aligned} &\left[\left(i\hbar\frac{d}{dt} - H\right)|\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle\right]\langle\widehat{\mu}_e©\widehat{\mu}_{n2}|\widehat{\mu}_e©\widehat{\mu}_{n2}\rangle = |\widehat{\mu}_e©\widehat{\mu}_{n1}\rangle \\ &\left(-i\hbar\frac{d}{dt}\langle\widehat{\mu}_e©\widehat{\mu}_{n2}| - \langle\widehat{\mu}_e©\widehat{\mu}_{n2}|H\right)|\widehat{\mu}_e©\widehat{\mu}_{n2}\rangle. \end{aligned} \tag{35}$$

From Eq. 30, we have

$$\langle\widehat{\mu}_e©\widehat{\mu}_{n2}|\widehat{\mu}_e©\widehat{\mu}_{n2}\rangle = C_{2+}^2\langle+z|+z\rangle + C_{2-}^2\langle-z|-z\rangle = C_{2+}^2 + C_{2-}^2 = 1. \tag{36}$$

Therefore, we reach

$$\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle=|\hat{\mu}_e©\hat{\mu}_{n1}\rangle\left(-i\hbar\frac{d}{dt}\langle\hat{\mu}_e©\hat{\mu}_{n2}|-\langle\hat{\mu}_e©\hat{\mu}_{n2}|H\right)|\hat{\mu}_e©\hat{\mu}_{n2}\rangle, \tag{37}$$

which can be written alternatively as

$$\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle=|\hat{\mu}_e©\hat{\mu}_{n1}\rangle\left[\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n2}\rangle\right]^{\dagger}|\hat{\mu}_e©\hat{\mu}_{n2}\rangle. \tag{38}$$

Because this equation holds for any two independent realizations, both sides must vanish, yielding

$$\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle=0 \tag{39}$$

and

$$\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n2}\rangle=0. \tag{40}$$

Ensemble averaging either equation over all co-quantum realizations produces the space-independent Schrödinger–Pauli equation,

$$\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e\rangle=0. \tag{41}$$

Conversely, we now assume that the *bra* and *ket* vectors originate from identical realizations and examine the consequence. Replacing each subscripted 2 with 1 in Eq. 37 yields

$$\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle=|\hat{\mu}_e©\hat{\mu}_{n1}\rangle\left(-i\hbar\frac{d}{dt}\langle\hat{\mu}_e©\hat{\mu}_{n1}|-\langle\hat{\mu}_e©\hat{\mu}_{n1}|H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle. \tag{42}$$

Replacing each subscripted 2 with 1 in Eq. 36 gives

$$\langle\hat{\mu}_e©\hat{\mu}_{n1}|\hat{\mu}_e©\hat{\mu}_{n1}\rangle=1. \tag{43}$$

Differentiation yields

$$\left(\frac{d}{dt}\langle\hat{\mu}_e©\hat{\mu}_{n1}|\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle+\langle\hat{\mu}_e©\hat{\mu}_{n1}|\left(\frac{d}{dt}|\hat{\mu}_e©\hat{\mu}_{n1}\rangle\right)=0. \tag{44}$$

Substitution into Eq. 42 gives

$$\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle=|\hat{\mu}_e©\hat{\mu}_{n1}\rangle\langle\hat{\mu}_e©\hat{\mu}_{n1}|\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle. \tag{45}$$

Replacing each subscripted 2 with 1 in Eq. 31 and substituting it into the above equation produces

$$\left(1-\rho_0\right)\left(i\hbar\frac{d}{dt}-H\right)|\hat{\mu}_e©\hat{\mu}_{n1}\rangle=0. \tag{46}$$

If the matrix, $1-\rho_0$, is full rank, multiplying both sides from the left by the inverse matrix uniquely yields the space-independent Schrödinger–Pauli equation. However, from Eq. 14, which holds for ρ_0 as well, we have the determinant, $\left|1-\rho_0\right|=0$; thus, the matrix is singular. Therefore, the space-independent Schrödinger–Pauli equation can be reached as only a sufficient but not necessary condition. To reach the Schrödinger–Pauli equation as a sufficient and necessary condition, we assume that the *bra* and *ket* vectors [14] originate from independent realizations, which is explicit here.

5 Discussion and Summary

Quantum mechanics, celebrated for its countless triumphs, still poses open questions as discussed insightfully in recent monographs [16–19]. Various thought experiments have been proposed [20–22]. The transition from classical physics to quantum mechanics remains an open question. In the Copenhagen interpretation, an electron spin is considered to be simultaneously in both eigenstates, and its wavefunction collapses statistically upon measurement to either eigenstate [12]. The collapse of wavefunction is stated separately as a measurement postulate because it cannot be modeled by the original Schrödinger equation [17].

We now extend the Bloch equation to the Landau–Lifshitz–Gilbert equation [23],

$$\frac{d\hat{\mu}}{dt} = \gamma\hat{\mu}\times\vec{B} - k_i\hat{\mu}\times\frac{d\hat{\mu}}{dt}. \tag{47}$$

Here, the dimensionless k_i is called the induction factor, which is used in CQD to explain the collapse of wavefunction (see Appendix). Although this equation was originally intended for condensed matter, the underlying physical mechanism for the added term is compatible with CQD. In fact, the author had developed CQD before realizing its connection with the Landau–Lifshitz–Gilbert equation. Setting $k_i=0$ recovers the Bloch equation.

Following the same procedure shown in Sect. 2, the Landau–Lifshitz–Gilbert equation (Eq. 47) can be converted to the following nonlinear variant of the von Neumann equation:

$$i\hbar\frac{d}{dt}\rho - \hbar k_i\left[\frac{d\rho}{dt},\rho\right] = [H,\rho]. \tag{48}$$

One may compare CQD with the existing quantum mechanical theories for collapse, e.g., the Ghirardi–Rimini–Weber model [24], continuous spontaneous localization model [25, 26], and the "Wavefunction Is the System Entity" (WISE) interpretation [27].

Attempted conversion to a variant of the Schrödinger–Pauli equation has yielded only

$$i\hbar\frac{d}{dt}\left|\hat{\mu}\right\rangle - \hbar k_i(1-\rho)\frac{d}{dt}\left|\hat{\mu}\right\rangle = H\left|\hat{\mu}\right\rangle, \tag{49}$$

where ρ cannot be eliminated yet. This situation may be related to the fact that the collapse of wavefunction cannot be modeled by the original Schrödinger equation [17].

In summary, the classical Bloch equation has been shown to lead to the space-independent von Neumann equation and the space-independent Schrödinger–Pauli equation, both for a pure state of electron spin. While it is known that the space-independent Schrödinger–Pauli equation or von Neumann equation for a unitary two-level system can be converted to the Bloch equation or its analog [11–13], the inverse conversion for electron spin is shown here. It is first shown that the Bloch equation and the space-independent von Neumann equation are equivalent for a pure state of electron spin. Further conversion from the space-independent von Neumann equation to the space-independent Schrödinger–Pauli equation as a both sufficient and necessary condition is proven under the assumption of independent realizations of the *bra* and *ket* vectors, which is implicit in quantum mechanics but explicit in CQD. Without such an assumption, the space-independent Schrödinger–Pauli equation is only a sufficient but not necessary condition to either the classical Bloch equation or the space-independent von Neumann equation. The presented transition from classical physics to quantum mechanics can potentially lead to new insight into some of the open questions.

Appendix: Co-quantum Dynamics

Because the manuscript on co-quantum dynamics (CQD) is being reviewed, an essential excerpt is provided below for completeness.

Abstract: In the classic multi-stage Stern–Gerlach experiment conducted by Frisch and Segrè, the Majorana (Landau–Zener) or Rabi formulae diverge afar from the experimental observation while the physical mechanism for electron-spin collapse remains unidentified. Here, introducing the physical co-quantum concept provides a plausible physical mechanism and predicts the experimental observation in absolute units without fitting (i.e., no parameters adjusted) highly accurately. Further, the co-quantum concept is corroborated by statistically reproducing exactly the wave function, density operator, and uncertainty relation for electron spin.

In typical Stern–Gerlach experiments, the dominant motion of $\hat{\mu}_e$ is precession about the main field, and the secondary motion is collapse due to induction—with the following trend:

$$\tan\frac{\theta_e(t)}{2} = \tan\frac{\theta_e(0)}{2}\exp\left[-\mathrm{sgn}\left(\theta_n - \theta_e\right)k_i\left|\Delta\phi_e(t)\right|\right]. \tag{50}$$

Here, $\Delta\phi_e$ denotes the traversed azimuthal angle (i.e., the phase). As time evolves, θ_e approaches either 0 or π according to the following branching condition:

$$\mathrm{sgn}\left(\theta_n - \theta_e\right) = \begin{cases} 1 & \text{if } \theta_n > \theta_e, \\ 0 & \text{if } \theta_n = \theta_e, \\ -1 & \text{else.} \end{cases} \tag{51}$$

Therefore, $\hat{\mu}_e$ collapses to either $+z$ or $-z$.

The CQD pre-collapse state function is denoted by $|\hat{\mu}_e©\hat{\mu}_n\rangle$, where the co-quantum, $\hat{\mu}_n$, is prefixed with © for clarity. $|\hat{\mu}_e©\hat{\mu}_n\rangle$ represents $\hat{\mu}_e$ accompanied with $\hat{\mu}_n$.

The CQD prediction expression for Stern–Gerlach experiments is written as

$$|\hat{\mu}_e©\hat{\mu}_n\rangle = C_+\left(\hat{\mu}_e, \hat{\mu}_n\right)|+z\rangle + C_-\left(\hat{\mu}_e, \hat{\mu}_n\right)\exp\left(i\phi_e\right)|-z\rangle. \tag{52}$$

The equal sign functions as a right arrow (→) because the right side predicts the measurement outcome. A given $\hat{\mu}_e$ collapses to either $+\hat{z}$ or $-\hat{z}$ according to the branching condition (Eq. 51). The two real and positive C coefficients take on mutually exclusive binary values while $\exp\left(i\phi_e\right)$ captures the phase information. If $\theta_n > \theta_e$, then $C_+ = 1$ and $C_- = 0$; if $\theta_n < \theta_e$, $C_+ = 0$ and $C_- = 1$. In either case, $C_+ \cdot C_- = 0$ and $C_+ + C_- = 1$.

Acknowledgements The author thanks Dr. Zhe He for discussing the manuscript and verifying the derivations and Prof. JT Shen for discussing the work.

Author Contributions NA (single author).

Funding No funding was received for conducting this study.

Data Availability All data used in this study are available from the author upon reasonable request.

Code Availability All custom codes used in this study are available from the author upon reasonable request.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Humans and/or Animals Rights NA (no human subjects/animals).

References

1. Field, J.: Derivation of the Schrödinger equation from the Hamilton-Jacobi equation in Feynman's path integral formulation of quantum mechanics. Eur. J. Phys. **32**(1), 63 (2010)

2. Scully, M.O.: The time dependent Schrodinger equation revisited I quantum field and classical Hamilton–Jacobi routes to Schrodinger's wave equation. J. Phys. **99**, 120 (2008)
3. Briggs, J.S., Boonchui, S., Khemmani, S.: The derivation of time-dependent Schrödinger equations. J. Phys. A: Math. Theor. **40**(6), 1289 (2007)
4. Hall, M.J., Reginatto, M.: Schrödinger equation from an exact uncertainty principle. J. Phys. A: Math. Gen. **35**(14), 3289 (2002)
5. Lamb, W.E.: Super classical quantum mechanics: The best interpretation of nonrelativistic quantum mechanics. Am. J. Phys. **69**(4), 413–422 (2001)
6. Lamb, W.E., Jr.: Suppose Newton had invented wave mechanics. Am. J. Phys. **62**(3), 201–206 (1994)
7. Schleich, W.P., Greenberger, D.M., Kobe, D.H., Scully, M.O.: Schrödinger equation revisited. Proc. Natl. Acad. Sci. **110**(14), 5374–5379 (2013)
8. Feynman, R.P.: Space-time approach to non-relativistic quantum mechanics. Rev. Mod. Phys. **20**, 367–387 (1948)
9. Majorana, E.: Atomi orientati in campo magnetico variabile. Il Nuovo Cimento (1924–1942) **9**(2), 43–50 (1932)
10. Majorana, E.: Oriented atoms in a variable magnetic field. In: Bassani, G. (ed.) Ettore Majorna: Scientific Papers, pp. 125–132. Società Italian die Fisica and Springer (2006)
11. Grynberg, G., Aspect, A., Fabre, C.: Introduction to quantum optics: from the semi-classical approach to quantized light. Cambridge University Press (2010)
12. Feynman, R.P., Leighton, R.B., Sands, M.L.: The Feynman Lectures on Physics. Addison-Wesley Pub. Co. (1963)
13. Feynman, R.P., Vernon, F.L., Jr., Hellwarth, R.W.: Geometrical representation of the Schrödinger equation for solving maser problems. J. Appl. Phys. **28**(1), 49–52 (1957)
14. Dirac, P.A.M.: The Principles of Quantum Mechanics. Oxford University Press, Oxford (1981)
15. Wieser, R.: Derivation of a time dependent Schrödinger equation as the quantum mechanical Landau–Lifshitz–Bloch equation. J. Phys.: Condens. Matter **28**(39), 396003 (2016)
16. Bricmont, J.: Making Sense of Quantum Mechanics. Springer, Berlin (2016)
17. Norsen, T.: Foundations of Quantum Mechanics. Springer, Berlin (2017)
18. Laloë, F.: Do We Really Understand Quantum Mechanics? Cambridge University Press, Cambridge (2019)
19. Auletta, G.: The Quantum Mechanics Conundrum: Interpretation and Foundations. Springer, Berlin (2019)
20. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete. Phys. Rev. **47**(10), 777–780 (1935)
21. Frauchiger, D., Renner, R.: Quantum theory cannot consistently describe the use of itself. Nat. Commun. **9**(1), 3711 (2018)
22. Schrödinger, E.: Die gegenwärtige situation in der quantenmechanik. Naturwissenschaften **23**, 807 (1935)
23. Gilbert, T.L.: A phenomenological theory of damping in ferromagnetic materials. IEEE Trans. Magn. **40**(6), 3443–3449 (2004)
24. Ghirardi, G.C., Rimini, A., Weber, T.: Unified dynamics for microscopic and macroscopic systems. Phys. Rev. D **34**(2), 470–491 (1986)
25. Pearle, P.: Combining stochastic dynamical state-vector reduction with spontaneous localization. Phys. Rev. A **39**(5), 2277 (1989)
26. Ghirardi, G.C., Pearle, P., Rimini, A.: Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles. Phys. Rev. A **42**(1), 78–89 (1990)
27. Long, G.: Collapse-in and collapse-out in partial measurement in quantum mechanics and its wise interpretation. Sci. China Phys. Mech. Astron. (2021). https://doi.org/10.1007/s11433-021-1781-x